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TORSIONAL VIBRATION IN AN ELASTIC ROD WITH EXTERNAL DRY FRICTION†

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The problem of forced oscillations in an elastic rod subject to harmonic excitation when there is dry external friction present, varying as given by Coulomb's law [1], is solved using approximate methods of expansion in a small parameter and harmonic linearization.

1. Consider a circular rod acted upon by a pressure that is constant along the length, loaded at one end with a torque which varies sinusoidally with frequency ω. The equation of the dynamics of such a rod has the form

$$\varphi^{\cdot\cdot} - c^2\varphi'' = q\,\mathrm{sign}\,\varphi^{\cdot}, \quad c^2 = G/\rho \tag{1.1}$$

Here $\varphi(x, t)$ is the angle of rotation of the cross-section with coordinate x, q is a coefficient which depends on the value of the uniformly distributed pressure on the rod and which characterizes the intensity of the torque, G is the shear modulus, and ρ is the density of the material. The primes and the dots denote partial derivatives with respect to the x coordinate and time t.

Considering the steady oscillations, the solution of Eq. (1.1) will be sought in the form

$$\begin{gathered}\varphi(x,t) = A_1 \sin\xi_1 + A_2 \sin\xi_2 \\ \xi_1 = \omega t + \alpha x + \varphi_1, \quad \xi_2 = \omega t - \alpha x - \varphi_2, \quad \alpha = \omega / c\end{gathered} \tag{1.2}$$

Solution (1.2) for constant A_1, A_2, φ_1 and φ_2 satisfies Eq. (1.1) with zero right-hand side. We will construct the solution of Eq. (1.1) by the method of varying arbitrary constants. Assuming functions of the coordinates to be constant and imposing the following conditions on the derivatives

$$A_1' \sin\xi_1 + A_1\varphi_1' \cos\xi_1 + A_2' \sin\xi_2 - A_2\varphi_2 \cos\xi_2 = 0 \tag{1.3}$$

we substitute the required solution (1.2) into (1.1) and, solving the system simultaneously with condition (1.3), we obtain the following system of non-linear equations for finding A_1, A_2, φ_1 and φ_2

$$A_j' = (-1)^{j+1} \frac{q}{\omega c} \cos\xi_j\, \mathrm{sign}\,\varphi^{\cdot}, \quad \varphi_j' = -\frac{q}{A_j \omega c} \sin\xi_j\, \mathrm{sign}\,\varphi^{\cdot}, \quad j = 1,2 \tag{1.4}$$

It is difficult to solve system (1.4) in general form. We will use the method of averaging. It was shown in [2, 3] that for weak damping in the case considered the oscillations can be assumed to be slowly varying with respect to both time and the coordinate. Hence, in the system of equations (1.4) we can carry out alternate averaging with respect to time and the coordinate. Introducing the function [3]

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$$|\varphi^{\cdot}| = |A_1\omega\cos\xi_1 + A_2\omega\cos\xi_2| \tag{1.5}$$

we can write the averaged system (1.4) over a period of time as

$$A_j' = \frac{(-1)^{j+1}}{2\pi}\int_0^{2\pi} \frac{q}{\omega c}\frac{\partial|\varphi^{\cdot}|}{\partial A_j\omega}d\omega t, \quad \varphi_j' = \frac{(-1)^{j+1}}{2\pi}\int_0^{2\pi} \frac{q}{A_j^2\omega^2 c}\frac{\partial|\varphi^{\cdot}|}{\partial\varphi_j}d\omega t, \quad j=1,2 \tag{1.6}$$

In system (1.6) we average under the differential sign

$$\int_0^{2\pi} |\varphi^{\cdot}|\, d\omega t = 4R, \quad \zeta = \psi_1 + \psi_2 \tag{1.7}$$

$$R = \left[A_1^2 + A_2^2 + 2A_1A_2\cos\zeta\right]^{1/2}, \quad \psi_j = y + \varphi_j$$

and change to dimensionless quantities. We will put $\alpha x = y$, $A_j = 4qa_j/(\pi^2\omega^2)$. Then, taking the averaging (1.7) into account, system (1.6) can be written in the form

$$a_j' = (-1)^{j+1}\frac{1}{2}\frac{\partial r}{\partial a_j}, \quad \varphi_j' = \frac{1}{2a_j^2}\frac{\partial r}{\partial\varphi_j} \tag{1.8}$$

$$r = \left(a_1^2 + a_2^2 + 2a_1a_2\cos\zeta\right)^{1/2}, \quad j = 1,2$$

We average over the coordinate $y = \alpha x$. To do this in system (1.8) we determine the mean value $\langle r\rangle$ over a period with respect to the fast variable $\zeta = \psi_1 = \psi_2$, where we assume that the quantities a_j and φ_j varies slowly. We obtain

$$\langle r\rangle = \frac{1}{2\pi}\int_0^{2\pi} r(a_1, a_2, \zeta)a\zeta = \frac{2(a_1+a_2)}{\pi}E(k), \quad k^2 = \frac{4a_1a_2}{(a_1+a_2)^2} \tag{1.9}$$

($E(k)$ is the complete elliptic integral of the second kind).

Substituting $\langle r\rangle$ into (1.8) we can write the averaged system in the form

$$a_j' = \frac{1}{\pi}\left\{(-1)^{j+1}E + \frac{E-K}{2a_j}(a_2 - a_1)\right\}, \quad \varphi_j' = 0, \quad j = 1,2 \tag{1.10}$$

($K(k)$ is the complete elliptic integral of the first kind).

We expand the elliptic integrals in series in powers of $\varepsilon = (1-k')/(1+k')$ where $k' = \sqrt{(1-k^2)}$ [4], and we substitute these expansions into (1.10), retaining terms of order not higher than ε. We then obtain the system

$$a_1' = 1/2, \quad a_2' = a_2/(4a_1), \quad \varphi_1' = \varphi_2' = 0 \tag{1.11}$$

System (1.11) has the solution

$$a_1 = \tfrac{1}{2}y + C_1, \quad a_2 = C_2/(\tfrac{1}{2}y + C_1)^{1/2}, \quad \varphi_1 = C_3, \quad \varphi_2 = C_4$$

Then, the required solution (1.2) can be represented in the initial variables in the form

$$D = \frac{4q}{\pi\omega^2}, \quad \operatorname{tg}\Psi = \frac{a_1\sin\psi_1 + a_2\sin\psi_2}{a_1\cos\psi_1 + a_2\cos\psi_2} \tag{1.12}$$

$$r=\left\{\left(\tfrac{1}{2}y+C_1\right)^2+C_2^2\left(\tfrac{1}{2}+C_1\right)^{-1}+2C_2\left(\tfrac{1}{2}y+C_1\right)^{\frac{1}{2}}\cos(2y+C_3+C_4)\right\}^{\frac{1}{2}}$$
$$y=\alpha x,\quad \alpha=\omega/c,\quad \psi_1=y+C_3,\quad \psi_2=y+C_4$$

The solution obtained agrees well with the numerical solution of the averaged system (1.8). We will compare the results of the solution of the averaged system (1.8) and of system (1.11) after averaging over the coordinate and expanding in terms of the small parameter.

To determine the constants of integration we will specify the boundary conditions.

We will assume that one end of the rod ($x=0$) is clamped, while a perturbing moment which varies sinusoidally with amplitude H and frequency ω acts on the other end at $x=L$. For simplicity we will take the kinematic specification of the perturbation. The boundary conditions for the dimensionless relationships can be written in the form

$$\begin{aligned}&\varphi(0,t)=0,\quad y=0\\&\varphi(l,t)=h\sin\omega t,\quad h=H/D,\quad \alpha L=l\\&\varphi(y,t)=a_1\sin(\omega t+y+C_3)+a_2\sin(\omega t-y-C_4)\end{aligned}\tag{1.13}$$

We will assume that torsional vibrations propagate over the whole length of the rod. We will obtain the parameters of the system and the external action giving rise to these conditions after determining the constants of integration.

The boundary conditions (1.13) lead to a system of equations for finding the constants of integration, from which we obtain

$$\begin{aligned}&C_2=-C_1^{\frac{3}{2}},\quad C_3=C_4,\quad \operatorname{tg}C_3=(a_{2l}+a_{1l})(a_{2l}-a_{1l})^{-1}\operatorname{tg}l\\&a_{2l}=C_1^{\frac{3}{2}}\left(\tfrac{1}{2}l+C_1\right)^{-\frac{1}{2}},\quad a_{1l}=\tfrac{1}{2}l+C_1\\&a_{1l}^2+C_1^3a_{1l}^{-1}-2a_{1l}^{\frac{1}{2}}C_1^{\frac{3}{2}}\cos 2l=h^2\end{aligned}\tag{1.14}$$

Here h is the dimensionless amplitude of the perturbing action, $l=\alpha L$ is the dimensionless length of the rod, and a_{jl} are the values of a_j when $y=l$.

The last relation in (1.14) enables us to find the constant C_1, and the remaining required constants of integration are expressed in terms of it.

We obtain the following expression for the dimensionless amplitude of the displacements

$$r\approx\tfrac{1}{2}y+C_1-C_1^{\frac{3}{2}}\left(\tfrac{1}{2}y+C_1\right)^{-\frac{1}{2}}\cos 2y\tag{1.15}$$

where the approximate equality applies in view of the smallness of the ratio $a_2/a_1=\varepsilon$.

Figure 1 shows $r(y)$ in dimensionless form. The continuous curves show the relations obtained from the approximate solution (1.12), and the dashed curves represent the results of numerical integration of system (1.8). The comparison was carried out for the same boundary conditions and a pressure that is constant over the length for several values of the amplitude h of the kinetic perturbing action on the end of the rod. It can be seen that averaging over the x coordinate and expansion in terms of the small parameter has only a slight effect on the form of the relationship between the amplitude and the coordinates compared with the results of numerical integration of the averaged system over time.

2. The solution (1.12) holds for short rods, in which the vibrations propagate over the whole length of the rod. We will determine the parameters of the system and the external action which ensure these conditions. It can be seen from (1.14) that when the amplitude of the perturbing action h is reduced the value of the constant C_1 approaches zero. In the limiting case when $C_1=0$, the solution for the amplitude is given by a linear function. In this case

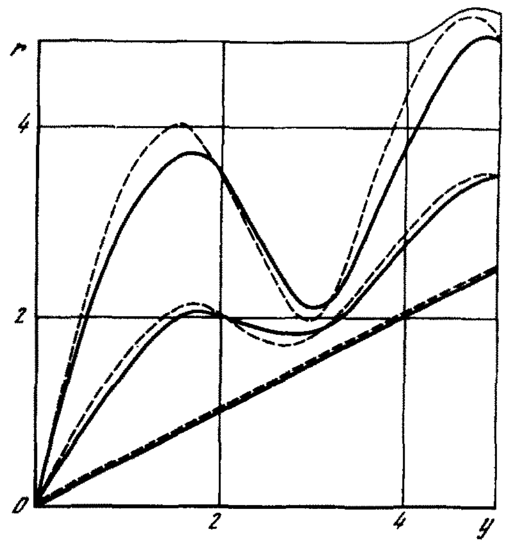

Fig. 1.

$$2h = l, \quad H\pi\omega^2/2 = \omega c^{-1}L \tag{2.1}$$

When $l = \alpha L$, as follows from the approximate solution, the vibrations will not reach the end of the rod. The limit of propagation of the vibration field can be established from condition (2.1), namely

$$L_* = H\pi\omega c/(2q) \leqslant L \tag{2.2}$$

We can then write for torsional vibrations of the cross sections of the rod for which condition (2.2) is satisfied

$$\varphi(x,t) = D\left(C_1 - \tfrac{1}{2}\alpha x\right)\sin\left(\omega t - \alpha x - \Psi_0\right) \tag{2.3}$$

Solution (2.3) identically satisfies system (1.8) if its solution is found in the form (1.2) by putting $A_2 = 0$. The constants of integration C_1 and $\Psi_0 = 0$ are found from the boundary conditions at the perturbed end. For example, for the kinematic boundary conditions $\varphi(0,\ t) = H\sin\omega t$ the constants of integration take the values $DC_1 = H$, $\Psi_0 = 0$, and the region of propagation of the vibrations x_* is found from the condition for the amplitude to be positive

$$x_* = 2C_1\alpha^{-1} = H\pi\omega c/(2q)$$

As $q \to 0$ the coordinate $x_* \to \infty$, which corresponds to the complete absence of damping. When the intensity of the torsional moment q increases, which depends on the forces of the distributed pressure, the vibrations are localized around the perturbed end of the rod.

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